PREDICTION OF SALT CONDITIONS IN IRRIGATED SOILS IN THE PRESENCE OF DRAINAGE

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The equations of convective diffusion are solved for two flat layers, using the method of finite integral transformations. Exact solutions are obtained to the problem in the presence of a descending or ascending filtration flow.

The washing of salty soils consists in feeding water to the surface of the massif; the water, filtering down, dissolves the salts and displaces them from the blanket layer into lower water tables with a deep bed of ground waters, or into drains and water collectors with an adjacent bed and a weak runoff of underground waters under natural conditions.

At the present time many specialists in land reclamation are convinced of the need for distilling not only the root-bearing layer of soil, but also the ground waters to a rather great depth, since distillation only of the upper layer of soil while retaining a high degree of mineralization of the adjacent ground waters is unstable, as a result of which washed soils frequently become salty at a high rate.

As is shown in a number of articles ([1], etc.), the zone of vertical salt transfer may reach 20-30 m, and in places 50-100 m, from the surface of the ground.

The present article considers the dynamics of salt solutions in a two-layer silt blanket formation. The upper layer lies above the level of the ground waters (the zone of incomplete saturation), while the lower layer lies between the ground waters and a roof consisting of a layer of sand, underlying the blanket silts.

It is postulated that the equation of convective diffusion holds both for the upper layer of silts and for the lower layer, which is completely saturated with water. It is assumed that the filtration and salt parameters for both strata are known as the results of experimental field investigations.

It is further assumed that drainage ensures a constant position of the level of the ground waters, and that the transport of salts takes place only in a vertical direction; this is confirmed in practice for a number of regions of the Central Asian and Caucasian Republics which are subject to salting; these regions have a clearly marked two-layer structure [2].

The mathematical problem comes down to solution of the following system of differential equations [2, 3]:

$$\frac{\partial C_i}{\partial t} = D_i \frac{\partial^2 C_i}{\partial x^2} - \frac{\varepsilon}{n_i} \frac{\partial C_i}{\partial x} + \gamma_i (C_* - C_i) \qquad (i = 1, 2) \tag{1}$$

with the initial and boundary conditions

$$C_i(x, 0) = \varphi_i(x) \tag{2}$$

$$\beta_1 C_1(0, t) - \alpha_1 \frac{\partial C_1(0, t)}{\partial x} = \beta_1 C_0 \tag{3}$$

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$$C_1(m_1, t) = C_2(m_1, t), \qquad \frac{\partial C_1(m_1, t)}{\partial x} = D_0 \frac{\partial C_2(m_1, t)}{\partial x} \tag{4}$$

$$\beta_2 C_2(m_2, t) + \alpha_2 \frac{\partial C_2(m_2, t)}{\partial x} = \beta_2 C_{00} \tag{5}$$

Here and in what follows the subscripts i = 1 and 2 refer, correspondingly, to the upper and lower layers of the blanket formation which is underlain by permeable sand stratum; $C_i(x, t)$ $\varphi_i(x)$ are the concentrations of the soil solution of salts at any arbitrary moment of time and at the initial moment, g/liter; C_*, C_0 are the concentrations of the limiting saturation of the given composition by salts, and of the water fed to washing, g/liter; D_i are the coefficients of filtrational diffusion (dispersion), m^2/day; α_i, β_i are constants whose values are given below during discussions of specific aspects of the problem which are of interest in land reclamation practice; γ_i are the coefficients of dissolution of the salts, day^{-1}; m_1, m_2 are, respectively, the depth of the bed of ground waters and the total thickness of both layers of the blanket formation, m; ε is the filtration rate, m/day; n_i are the porosities of the layers; $C_{00}(t)$ is the mineralization of the ground waters in an easily permeable underlying stratum, a known function of time, g/liter; x is a coordinate, m; the x axis is directed vertically downwards, and the origin of coordinates is taken at the surface of the gound; t is thetime (days); $D_0 = n_2D_2/n_1D_1$.

By analogy with the usual Koshlyakov–Greenberg method of integral transformations [4, 5], we multiply Eqs. (1) and (2), respectively, by the kernal $K_i(x, \rho)$ and integrate twice by parts within the appropriate limits. As a result, in place of the starting system we obtain the ordinary system of equations

$$\partial U_i / \partial t + \rho^2 U_i = H_i + F_i \qquad (i = 1, 2) \tag{6}$$

where

$$U_i(t, \rho) = \int_{m_{i-1}}^{m_i} C(x, t) K_i(x, \rho)\, dx, \quad F_1(\rho) = \gamma_1 C_* \int_0^{m_1} K_1(x, \rho)\, dx, \qquad F_2(\rho) = 0$$

$$H_i(t, \rho) = \left[\left(D_i \frac{\partial C_i}{\partial x} - \frac{\varepsilon}{n_i} \right) K_i - C_i D_i \frac{dK_i}{dx} \right] \Bigg|_{m_{i-1}}^{m_i} \qquad (m_0 = 0)$$

We postulate that, in the lower layer (below the level of the ground waters), there are no salts in the solid phase ($\gamma_2 = 0$).

To find the kernal of the transformation we obtain the system of equations

$$D_i \frac{d^2K_i}{dx^2} + \frac{\varepsilon}{n_i} \frac{dK_i}{dx} + (\rho^2 - \delta_i) K_i = 0 \qquad (i = 1, 2) \tag{7}$$

$$(\delta_1 = \gamma_1, \qquad \delta_2 = \gamma_2 = 0)$$

We set the functions $K_i(x, \rho)$ equal to $M^{-1}(\rho) N_i(x, \rho)\ \sigma_i(x)$, where M^{-1} is a normalizing divisor, and $N_i(x, \rho)$ and $\sigma_i(x)$ are unknown functions, determined, respectively, from the equations

$$D_i \frac{d\sigma_i}{dx} + \frac{\varepsilon}{n_i} \sigma_i = 0 \tag{8}$$

$$D_i \frac{d}{dx} \left(\sigma_i \frac{dN_i}{dx} \right) + (\rho^2 - \delta_i) N_i = 0$$

Equations (8) are obtained after substitution of the values of $K_i(x, \rho)$ into (7). From the last two equations, taking account of the equality $\sigma_1(m_1) = \sigma_2(m_1)$, we find the functions

$$\sigma_1(x) = \exp\left(-\frac{x\varepsilon}{n_1 D_1} \right), \qquad \sigma_2(x) = \exp\left[-\frac{(x - m_1)\varepsilon}{n_2 D_2} - \frac{m_1 \varepsilon}{n_1 D_1} \right] \tag{9}$$

Determination of $N_1(x, \rho)$ and $N_2(x, \rho)$ reduces to the solution of the Sturm-Liouville problem with homogeneous boundary conditions.

Substituting (9) into (8) we obtain the equations

$$D_i \frac{d^2N_i}{dx^2} - \frac{\varepsilon}{n_i} \frac{dN_i}{dx} + (\rho^2 - \delta_i) N_i = 0 \qquad (i = 1, 2) \tag{10}$$

which are solved with the boundary conditions (3)-(5) with zero right-hand parts.

The general solution of system (1) has the form

$$N_i(x, \rho) = \exp(\tau_i x)[A_i \sin a_i\tau + B_i \cos a_i\tau] \tag{11}$$

From the homogeneous boundary conditions (3)-(5) (with $C_0 = C_{00} = 0$) we obtain a system of four equations for determining the coefficients A_i and B_i. For this homogeneous system of equations to have a nontrivial solution it is necessary and sufficient that the determinant of this system be equal to zero, $\Delta(\rho) = 0$.

We write an expression for this determinant

$$\Delta(\rho) = \mathrm{tg}\,[a_2(m_2 - m_1)] - \frac{\theta_4 - \theta_3 \,\mathrm{tg}\, a_1 m_1}{\theta_3 + \theta_1 \,\mathrm{tg}\, a_1 m_1} = 0 \tag{12}$$

and for the coefficients A_i, B_i

$$\begin{aligned} A_1 &= -a_{02}(b_{01}b_{12} - b_{02}b_{11}), \qquad B_1 = -A_1 a_{01}/a_{02} \\ A_2 &= -b_{02}(a_{01}a_{12} - a_{02}a_{11}), \qquad B_2 = -A_2 b_{01}/b_{02} \end{aligned} \tag{13}$$

Here we introduce the following notation:

$$a_{01} = \alpha_1 a_1, \qquad b_{01} = \alpha_1(\tau_2 \sin a_2 m_2 + a_2 \cos a_2 m_2) + \beta_2 \sin a_2 m_2$$
$$a_{02} = \alpha_1 \tau_1 - \beta_1, \qquad b_{02} = \alpha_2(\tau_2 \cos a_2 m_2 - a_2 \sin a_2 m_2) + \beta_2 \cos a_2 m_2$$
$$a_{11} = \exp(\tau_1 m_1) \sin a_1 m_1, \qquad b_{11} = \exp(\tau_2 m_1) \sin a_2 m_1$$
$$a_{12} = \exp(\tau_1 m_1) \cos a_1 m_1 \qquad b_{12} = \exp(\tau_2 m_1) \cos a_2 m_1$$
$$a_i^2 = \frac{1}{D_0}(\rho^2 - \delta_i) - \tau_i^2, \qquad \tau_i = \frac{\varepsilon}{2n_i D_i} \qquad (i = 1, 2)$$
$$\theta_1 = p_1(\beta_1 - \alpha_1\tau_1) + p_2(\beta_2 - \alpha_2\tau_2), \qquad \theta_2 = a_1(\alpha_1 p_1 - \beta_1\beta_2 - \beta_1\alpha_2\tau_2)$$
$$\theta_3 = a_2(\alpha_2 p_2 + \beta_1\beta_2 D_0 - \alpha_1\beta_2\tau_1 D_0), \qquad \theta_4 = a_1 a_2(\alpha_1\beta_2 D_0 + \alpha_2\beta_1)$$
$$p_1 = \alpha_2 D_0(\tau_2^2 + a_2^2) + \beta_2\tau_2 D_0, \qquad p_2 = \tau_1\beta_1 - \alpha_1(\tau_1^2 + a_1^2)$$

Thus, the functions $N_i(x, \rho)$ (11) are the solution of the Sturm-Liouville problem whose eigenvalues are found as the roots of the transcendental equation (12).

Let us now consider solution of system (6) for the transforms. We multiply these equations respectively by n_i and, combining, we have

$$\partial W/\partial t + \rho^2 W = H(t, \rho) + n_1 F_1(\rho) \tag{14}$$

where the following notation is introduced:

$$W(t, \rho) = n_1 U_1(t, \rho) + n_2 U_2(t, \rho), \; H(t, \rho) = n_1 H_1 + n_2 H_2$$

Using the boundary conditions for the sought functions C_i and N_i and the equality $\sigma_1(m_1) = \sigma_2(m_1)$, in the expression for $H(t, \rho)$ we can eliminate the boundary conditions at the point of contact between the layers. As a result we obtain

$$H(t, \rho) = \frac{1}{M(\rho)}\left[\frac{1}{\alpha_1} n_1 D_1 \beta_1 C_0 N_1(0, \rho) + \frac{1}{\alpha_2} n_2 D_2 \beta_2 \sigma_2(m_2) C_{00} N_2(m_2, \rho)\right] \tag{15}$$

We take the initial conditions (3) in the form

$$W(0, \rho) = n_1 U_1(0, \rho) + n_2 U_2(0, \rho) \tag{16}$$

where $U_i(0, \rho)$ are the values of the transforms of the functions $\varphi_i(x)$.

The solution of Eq. (14) with the condition (16) is easily found; in particular, with constant values of C_0 and C_{00}, it has the form

$$W(t, \rho) = \frac{1}{\rho^2}[H(\rho) + n_1 F_1(\rho)] + \left[W(0, \rho) - \frac{H(\rho) + n_1 F_1(\rho)}{\rho^2}\right] \exp(-\rho^2 t) \tag{17}$$

We seek the inversion formulas for the functions $C_i(x, t)$ in the form of series in terms of the eigenfunctions of the Sturm-Liouville problem [4, 5]

$$C_i(x, t) = \sum_{\nu=1}^{\infty} S_\nu(t, \rho_\nu) N_i(x, \rho_\nu) \qquad (i = 1, 2) \tag{18}$$

where ρ_ν are the eigenvalues of the problem.

TABLE 1

x, m	ν				
	1	3	5	7	10
0	$\frac{0.950}{0.508}$	$\frac{2.929}{0.556}$	$\frac{3.611}{0.777}$	$\frac{3.818}{0.908}$	$\frac{3.932}{1.014}$
1	$\frac{24.516}{4.565}$	$\frac{31.614}{5.569}$	$\frac{22.199}{5.714}$	$\frac{21.053}{5.635}$	$\frac{20.213}{5.782}$

We multiply expressions (18), correspondingly, by $n_i K_i(x, \rho_j)$, integrate within the appropriate limits, and combine. As a result we obtain

$$W(t, \rho_j) = \sum_{\nu=1}^{\infty} \frac{S_\nu(t, \rho_\nu)}{M(\rho_j)} \left[n_1 \int_0^{m_1} \sigma_1(x) N_1(x, \rho_\nu) N_1(x, \rho_j)\, dx + n_2 \int_{m_1}^{m_2} \sigma_2(x) N_2(x, \rho_\nu) N_2(x, \rho_j)\, dx \right]$$

It is easily shown that, by virtue of the orthogonality of the functions N_1 and N_2, the expression in square brackets will be equal to zero at $\rho_\nu \neq \rho_j$ ($\nu \neq j$) while, at $\rho_\nu = \rho_j$ ($\nu = j$) it is equal to the value of M, if it is set equal to

$$M(\rho_\nu) = n_1 \int_0^{m_1} \sigma_1(x) N_1^2(x, \rho_\nu)\, dx + n_2 \int_{m_1}^{m_2} \sigma_2(x) N_2^2(x, \rho_\nu)\, dx \qquad (19)$$

It follows from this that $S_\nu(t, \rho_\nu) \equiv W(t, \rho_\nu)$ and the inversion formulas have the form (18), in which S_ν must be replaced by $W(t, \rho_\nu)$.

Thus, the series (18), together with (11)-(13), (17), and (19), yield the solution of the problem under consideration.

The following partial cases of the general problem are of interest:

1. The case $\alpha_2 = 0$ (or $C_2(m_2, t) = C_{00}$) and $\alpha_1 = n_1 D_1$, $\beta_1 = \varepsilon$. From (12) we have

$$\frac{\tau_1 a_2}{\tau_2 a_1} \operatorname{ctg}[a_2(m_2 - m_1)] = \frac{a_1 \operatorname{tg} a_1 m_1 - \tau_1}{\tau_1 \operatorname{tg} a_1 m_1 + a_1} \qquad (20)$$

For the function $H(t, \rho_\nu)$, we obtain the expression

$$H(t, \rho_\nu) = \frac{1}{M(\rho_\nu)} \left[\varepsilon C_0 N_1(0, \rho_\nu) - n_2 D_2 C_{00} \sigma_2(m_2) \frac{\partial N_2(m_2, \rho_\nu)}{\partial x} \right] \qquad (21)$$

2. The case $\beta_2 = 0$ (or $\partial C_2(m_2, t)/\partial x = 0$), $\alpha_1 = n_1 D_1$.

Then the characteristic equation and the functions $H(t, \rho_\nu)$ assume the form

$$\operatorname{tg}[a_2(m_2 - m_1)] = \frac{2\tau_2 a_1 a_2 + (\tau_1 \tau_2 a_2 - a_1^2 a_2 D_0^{-1}) \operatorname{tg} a_1 m_1}{a_1 a_2^2 - a_1 \tau_2^2 + (a_2^2 \tau_1 + a_1^2 \tau_2 D_0^{-1}) \operatorname{tg} a_1 m_1} \qquad (22)$$

$$H(t, \rho_\nu) = \varepsilon C_0 N_1(0, \rho_\nu) M^{-1}(\rho_\nu)$$

3. The case of the salting of the aeration zone of the blanket layer, with the evaporation of underground waters. In this case, in the solutions for the partial cases 1 and 2, we must set $C_0 = 0$ and replace the sign of the rate ε by the opposite value.

In a consideration of a single-layer blanket formation of silts, in all the preceding expressions we must set

$$m_1 = m_2 = m, \ \sigma_1 = \sigma_2 = \sigma(x), \ \tau_1 = \tau_2 = \tau, \ a_1 = a_2 = a, \ D_0 = 1$$

Under these circumstances the solutions are simplified and the solution reduces to the application of a finite integral Koshlyakov–Greenberg transform [4].

Calculations in accordance with the formulas obtained can be carried out in small electronic computers.

As an illustration, a calculation was made of the redistribution of the salts in a soil with washing, for the first case ($\alpha_2 = 0$), using the following data:

$$D_1 = 0.002\ \mathrm{m^2/day},\ D_2 = 0.02\ \mathrm{m^2/day},\ n_1 = 0.2,\ n_2 = 0.5$$
$$\varepsilon = 0.002\ \mathrm{m/day},\ m_1 = 2\ \mathrm{m},\ m_2 = 10\ \mathrm{m}$$
$$C_1(x, 0) = C_2(x, 0) = 20\ \mathrm{g}/l,\ \gamma = 0$$

The calculation was made in a "Nairi" electronic computer under automatic programming conditions. In the sums of formulas (18), a different number of terms of the series was taken $\nu = 1, 3, 5, 7, 10, 15$, and the sought functions were found. The roots ρ_ν were determined from Eq. (20).

Table 1 gives values of the concentrations of the salts, $C_1(x, t)$, in the upper layer in the cross sections $x = 0$ and 1 m for the moments of time $t = 60$ days (in the numerator) and $t = 200$ days (in the denominator).

An analysis of the solution of the example under consideration shows that, for a practical degree of accuracy, in the sums of (18) it is sufficient to retain 10 terms. With an increase in the Fourier numbers F_0 ($F_0 = D_1 t / m_1^2$), a lesser number of terms may be retained in the sums. Thus, with $F_0 \geq 0.1$, only the first three to five terms need be taken into consideration.

The proposed method of integrating the differential equations permits broadening the solution of a number of problems of mass transfer in porous media, and, in particular, examining the dynamics of salts in multilayer soils. In the latter case, it is useful to use a matrix description of the operations.

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